Section 3.4 Curve sketching

1) $f(x) = x^3 - 3x^2$

1a) Find the x-intercept(s), if any

$$x^3 - 3x^2 = 0$$
$$x^2(x-3) = 0$$
$$x \cdot x(x-3) = 0$$
$$x = 0 \quad x = 0 \quad x-3 = 0$$
$$x = 3$$

X-intercepts
$(0,0) \ (3,0)$

1b) Find the y-intercept, in there is one

$$y = f(0) = (0)^3 - 3(0)^2 = 0$$
$$y \text{ int } (0,0)$$

1c) Find the interval(s) where the graph of the function is increasing
1d) Find the interval(s) where the graph of the function is decreasing
1e) Find all relative maxima
1f) Find all relative minima

$$f(x) = x^3 - 3x^2$$

$$f'(x) = 3x^2 - 6x$$

$$3x^2 - 6x = 0$$

$$3x(x-2) = 0$$

$3x = 0$ $x - 2 = 0$

$x = 0$ $x = 2$

$(-\infty, 0)$ $(0, 2)$ $(2, \infty)$

$x = -1$ $x = 1$ $x = 3$

$f'(-1) = 3(-1)(-1-2)$ $f'(1) = 3(1)(1-2)$ $f'(3) = 3(3)(3-2)$

$= 9$ $= -3$ $= 9$

Increasing   Decreasing   Increasing
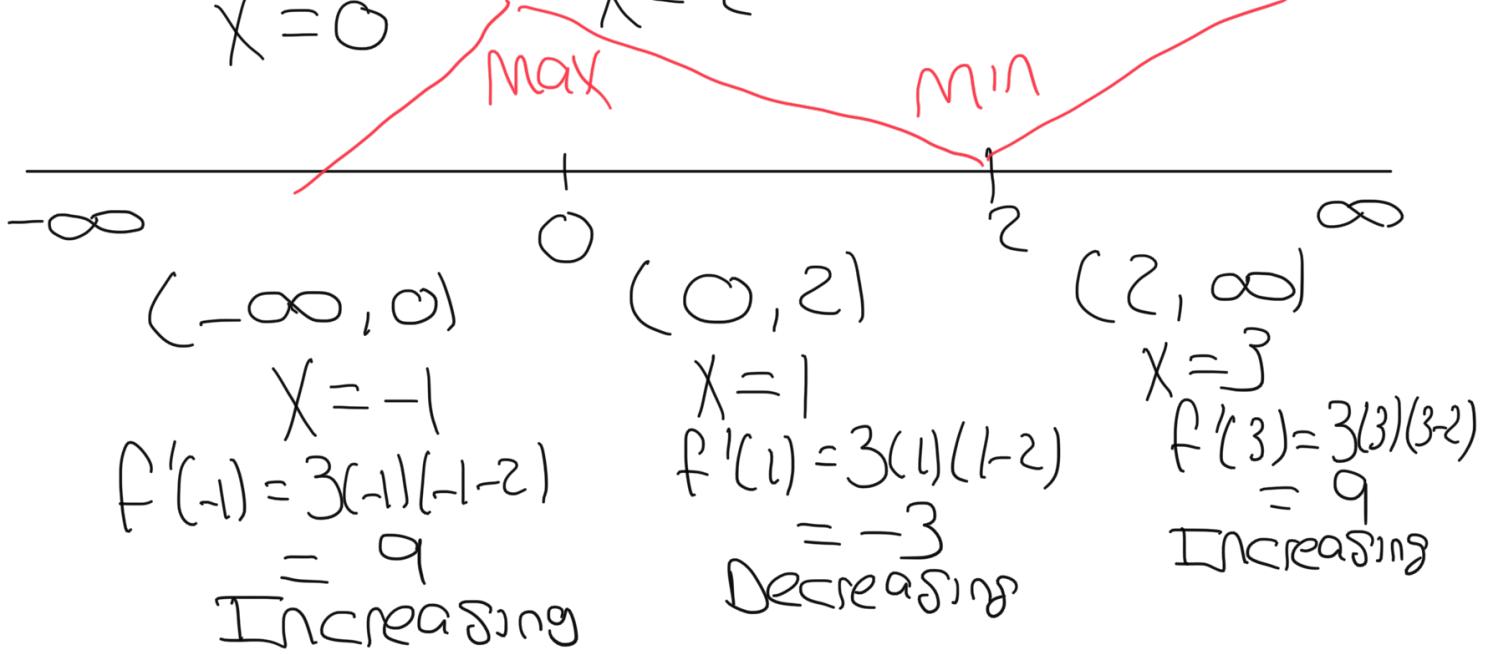
Max $x = 0$

$y = f(0) = (0)^3 - 3(0)^2 = 0$

Max

$(0, 0)$

Min $x = 2$

$y = f(2) = (2)^3 - 3(2)^2 = -4$

Min $(2, -4)$

1g) Find the interval(s) where the graph of the function is concave up (if any)
1h) Find the interval(s) where the graph of the function is concave down (if any)
1i) Find all inflection points (if any)
1j) Sketch a graph

$$f(x) = x^3 - 3x^2$$
$$f'(x) = 3x^2 - 6x$$
$$f''(x) = 6x - 6$$
$$6x - 6 = 0$$
$$6x = 6$$
$$x = 1$$

$$(-\infty, 1) \qquad (1, \infty)$$

$-\infty$ ——————— | ——————— ∞

$$x = 0$$
$$f''(0) = 6(0) - 6$$
$$= -6$$
Concave
Down

$$x = 2$$
$$f''(2) = 6(2) - 6$$
$$= 6$$
Concave
Up
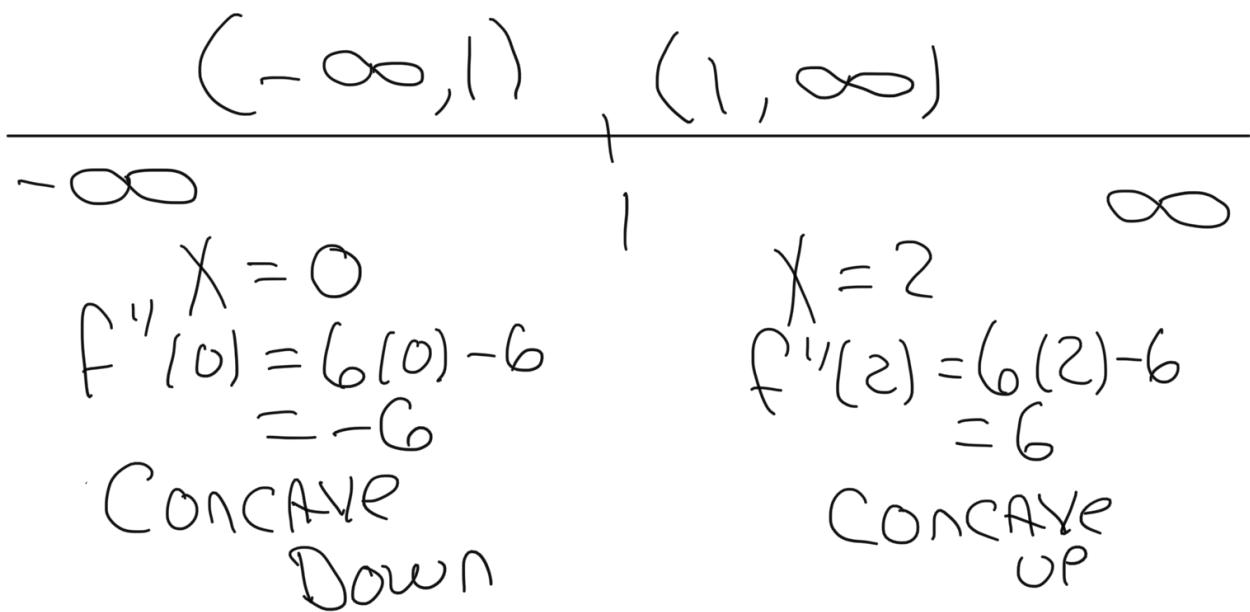
$$x\text{-coord Inflection point } x = 1$$
$$y = f(1) = (1)^3 - 3(1)^2 = -2$$

Inflection point $(1, -2)$

1a) Find the x-intercept(s), if any $(0,0)$ and $(3,0)$
1b) Find the y-intercept, in there is one $(0,0)$
1c) Find the interval(s) where the graph of the function is increasing $(-\infty, 0) \cup (2, \infty)$
1d) Find the interval(s) where the graph of the function is decreasing $(0,2)$
1e) Find all relative maxima $(0,0)$
1f) Find all relative minima $(2, -4)$
1g) Find the interval(s) where the graph of the function is concave up (if any) $(1, \infty)$
1h) Find the interval(s) where the graph of the function is concave down (if any) $(-\infty, 1)$
1i) Find all inflection points (if any) $(1, -2)$
1j) Sketch a graph
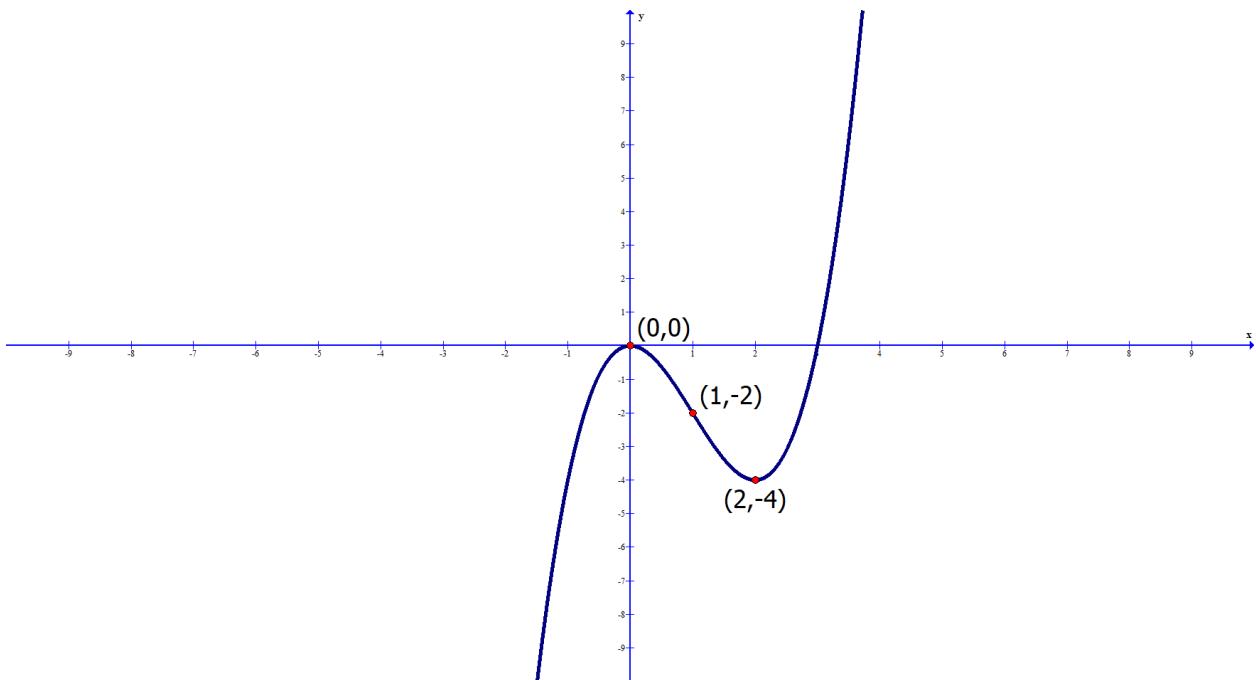
3) $f(x) = x^3 - 27x$

3a) Find the x-intercept(s), if any

$x(x^2 - 27) = 0$

$x = 0 \qquad x^2 - 27 = 0$

$\sqrt{x^2} = \pm\sqrt{27}$

$x = \pm\sqrt{27}$

or $\pm 3\sqrt{3}$

$(0, 0)$

$(\sqrt{27}, 0)$ OR $(3\sqrt{3}, 0)$

$(-\sqrt{27}, 0)$ or $(-3\sqrt{3}, 0)$

X-INT

3b) Find the y-intercept, in there is one

$y = f(0) = 0^3 - 27(0)$

$= 0$

Y-INT

$(0, 0)$

3c) Find the interval(s) where the graph of the function is increasing
3d) Find the interval(s) where the graph of the function is decreasing
3e) Find all relative maxima
3f) Find all relative minima

$$f(x) = x^3 - 27x$$
$$f'(x) = 3x^2 - 27$$
$$3(x^2 - 9) = 0$$
$$3(x+3)(x-3) = 0$$

No Solution

$$X = -3 \qquad X = 3$$

max   min

$-\infty$

$(-\infty, -3)$ -3 $(-3, 3)$ 3 $(3, \infty)$

$X = -4$ $X = 0$ $X = 4$

$$f'(-4) = 3(-4)^2 - 27 \qquad f'(0) = 3(0)^2 - 27 \qquad f'(4) = 3(4)^2 - 27$$
$$= 21 \qquad\qquad = -27 \qquad\qquad\qquad = 21$$

Increasing   DEcreasing   Increasing
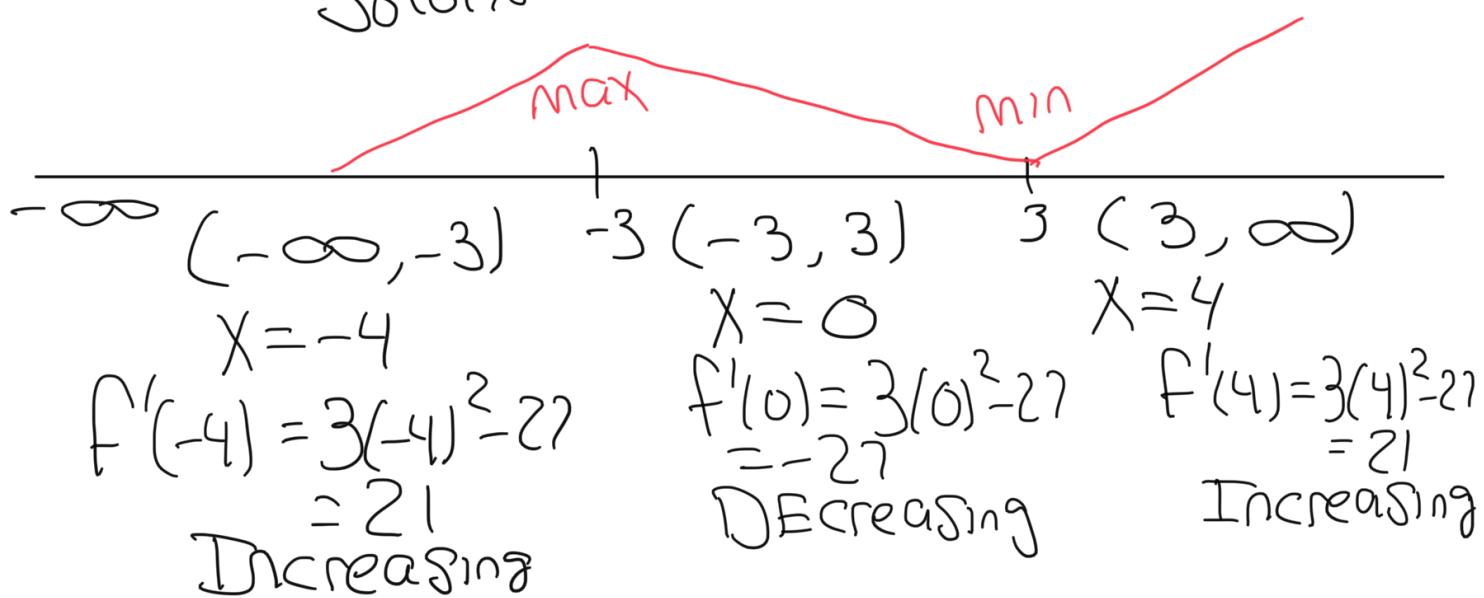
Xcoord max $X = -3$
$$y = f(-3) = (-3)^3 - 27(-3) = 54$$

Xcoord min $X = 3$
$$y = f(3) = (3)^3 - 27(3)$$

3g) Find the interval(s) where the graph of the function is concave up (if any)
3h) Find the interval(s) where the graph of the function is concave down (if any)
3i) Find all inflection points (if any)
3j) Sketch a graph

$$f(x) = x^3 - 27x$$
$$f'(x) = 3x^2 - 27$$
$$f''(x) = 6x$$

$$\frac{6x}{6} = \frac{0}{6}$$

$$x = 0$$

$(-\infty, 0)$

$x = -1$

$$f''(-1) = 6(-1)$$
$$= -6$$

CONCAVE
Down

$(0, \infty)$

$x = 1$

$$f''(1) = 6(1) = 6$$

CONCAVE UP
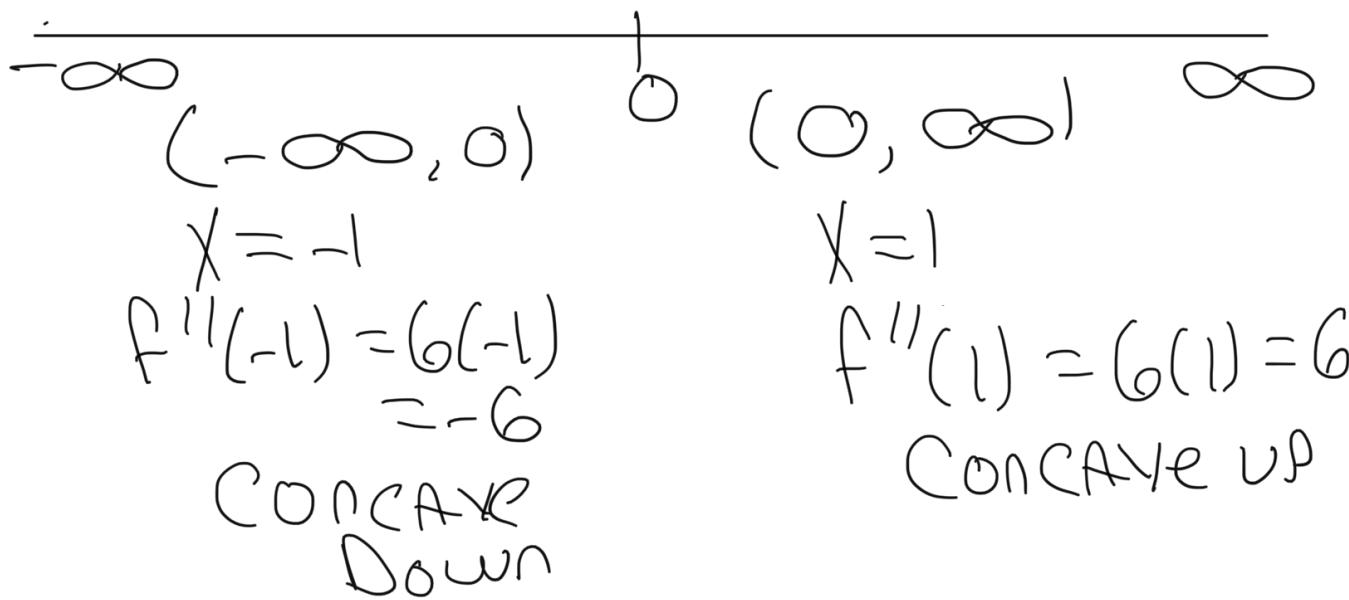
x-coordinate $x = 0$
Inflection point

$$y = (0)^3 - 27(0) = 0$$

Inflection POINT $(0, 0)$

or $(3\sqrt{3}, 0)\ (-3\sqrt{3}, 0)$

3a) Find the x-intercept(s), if any $(0,0)\ and\ (\sqrt{27}, 0)\ (-\sqrt{27}, 0)$

3b) Find the y-intercept, in there is one $(0,0)$

3c) Find the interval(s) where the graph of the function is increasing $(-\infty, -3) \cup (3, \infty)$

3d) Find the interval(s) where the graph of the function is decreasing $(-3,3)$

3e) Find all relative maxima ~~(-3, 54)~~ $(-3, 54)$

3f) Find all relative minima $(3, -54)$

3g) Find the interval(s) where the graph of the function is concave up (if any) $(0, \infty)$

3h) Find the interval(s) where the graph of the function is concave down (if any) $(-\infty, 0)$

3i) Find all inflection points (if any) $(0,0)$

3j) Sketch a graph
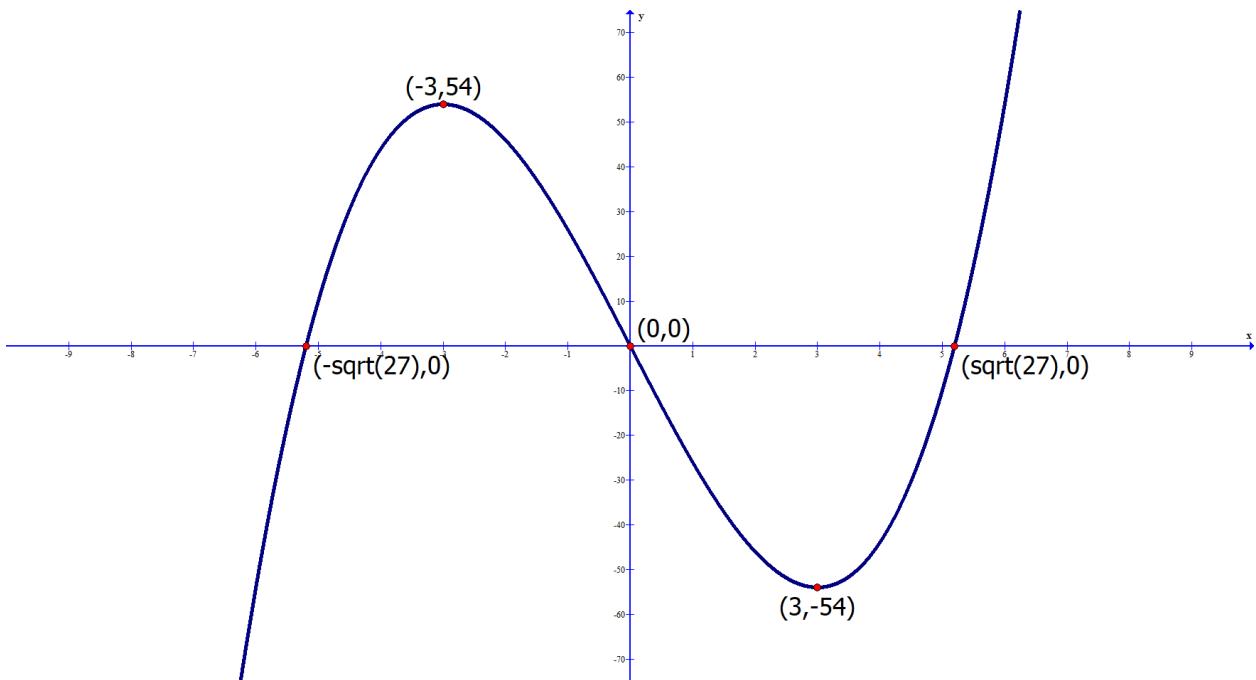
5) $f(x) = x^4 - 4x^3$

5a) Find the x-intercept(s), if any

$$x^4 - 4x^3 = 0$$
$$x^3(x-4) = 0$$
$$x \cdot x \cdot x = 0 \qquad x - 4 = 0$$
$$x = 0 \qquad x = 0 \quad x = 0 \qquad x = 4$$

X-INTERCEPTS
$(0,0)$
$(4,0)$

5b) Find the y-intercept, in there is one

$$y\text{-INT} \quad (0,0)$$

$$y = f(0) = (0)^4 - 4(0)^3$$
$$= 0$$

5c) Find the interval(s) where the graph of the function is increasing
5d) Find the interval(s) where the graph of the function is decreasing
5e) Find all relative maxima
5f) Find all relative minima

$$F(x) = X^4 - 4X^3$$
$$F'(x) = 4X^3 - 12X^2$$
$$4X^3 - 12X^2 = 0$$
$$4X^2(X-3) = 0$$

$$\frac{4X \cdot X = 0}{4} \qquad X-3 = 0$$
$$X \cdot X = 0 \qquad X = 3$$
$$X = 0$$

min

$-\infty \qquad 0 \qquad 3 \qquad \infty$

$(-\infty, 0) \qquad (0,3) \qquad (3,\infty)$
$X = -1 \qquad X = 1 \qquad X = 4$
$f'(-1) = 4(-1)^3 - 12(-1)^2 \qquad \qquad f'(4) = 4(4)^3 - 12(4)^2$
$\quad = -16 \qquad f'(1) = 4(1)^3 - 12(1)^2 \qquad = 64$
DECREASING $\qquad = -8 \qquad$ Increasing
$\qquad \qquad$ DECREASING
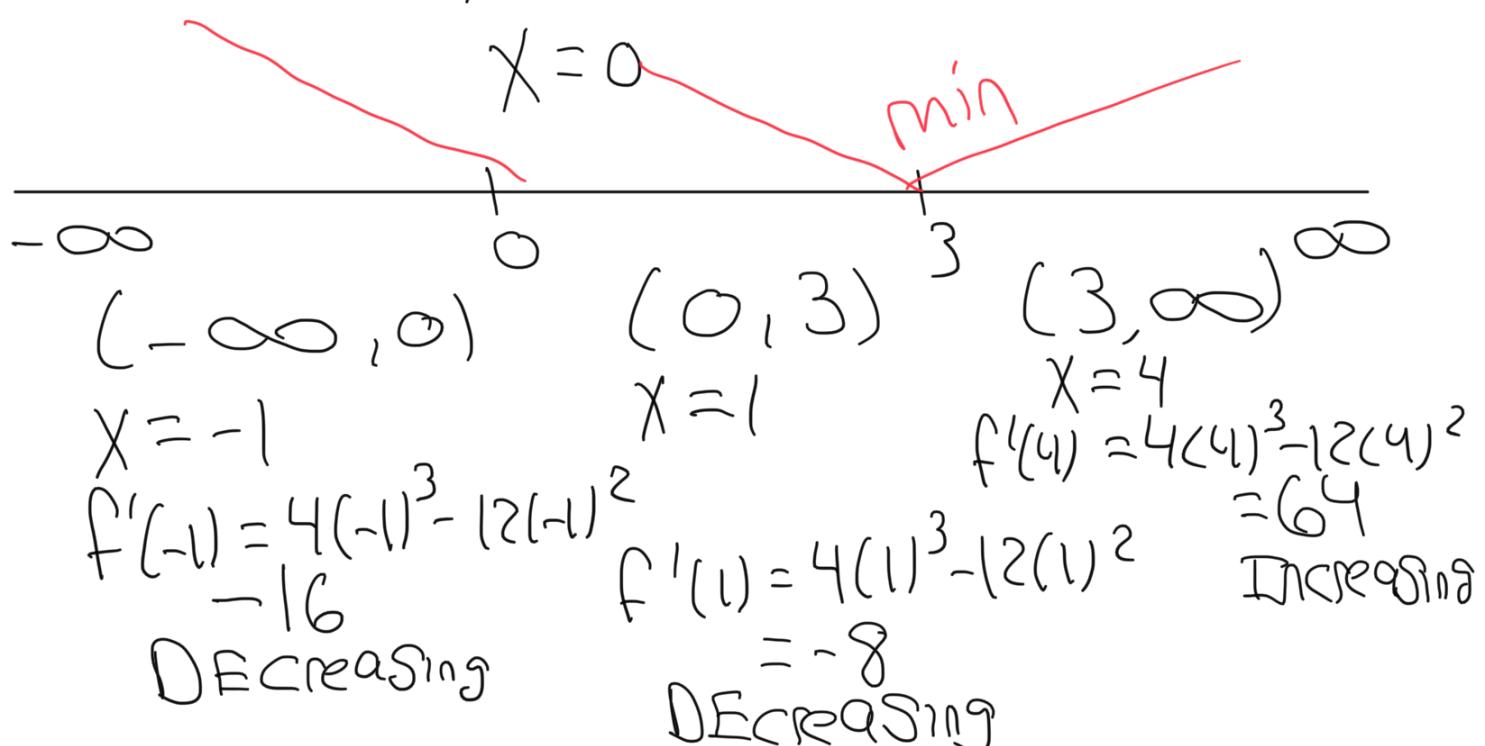
no max
X coord Min X=3
$$y = f(3) = (3)^4 - 4(3)^3 = -27$$
Min $(3, -27)$

5g) Find the interval(s) where the graph of the function is concave up (if any)
5h) Find the interval(s) where the graph of the function is concave down (if any)
5i) Find all inflection points (if any)

$$f(x) = x^4 - 4x^3$$
$$f'(x) = 4x^3 - 12x^2$$
$$f''(x) = 12x^2 - 24x$$

$$12x^2 - 24x = 0$$
$$12x(x-2) = 0$$

$$12x = 0 / 12 \qquad x - 2 = 0$$
$$\frac{12x}{12} \qquad x = 2$$

$$x = 0$$

$-\infty$ ———— 0 ———— 2 ———— ∞

$(-\infty, 0)$ $(0, 2)$ $(2, \infty)$

$x = -1$ $x = 1$ $x = 3$

$f''(-1) = 12(-1)^2 - 24(-1)$ $f''(1) = 12(1)^2 - 24(1)$ $f''(3) = 12(3)^2 - 24(3)$

$= 36$ $= -12$ $= 36$

Concave Up   Concave Down   Concave Up
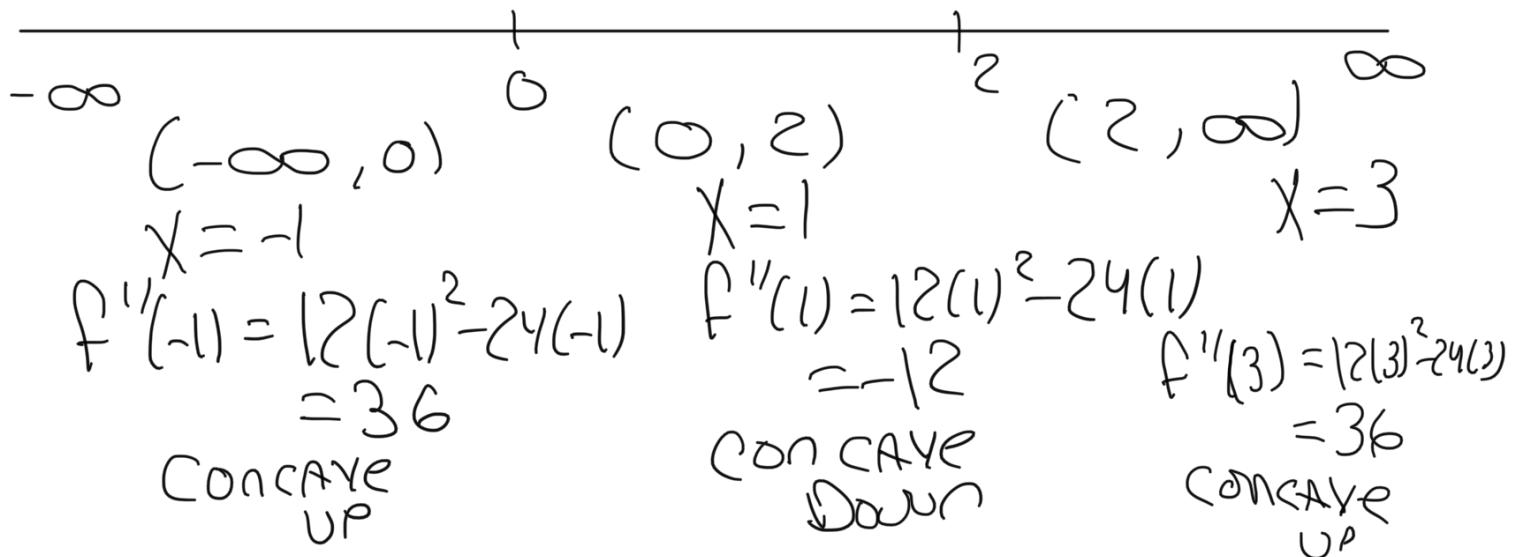
2 Inflection Points

$x = 0$ $y = f(0) = (0)^4 - 4(0)^3 = 0$ $(0, 0)$

$x = 2$ $y = f(2) = (2)^4 - 4(2)^3 = -16$ $(2, -16)$

5a) Find the x-intercept(s), if any $(0,0)\ and\ (4,0)$

5b) Find the y-intercept, in there is one $(0,0)$

5c) Find the interval(s) where the graph of the function is increasing $(3,\infty)$

5d) Find the interval(s) where the graph of the function is decreasing $(-\infty,0)\cup(0,3)$

5e) Find all relative maxima *none*

5f) Find all relative minima $(3,-27)$

5g) Find the interval(s) where the graph of the function is concave up (if any) $(-\infty,0)\cup(2,\infty)$

5h) Find the interval(s) where the graph of the function is concave down (if any) $(0,2)$

5i) Find all inflection points (if any) $(0,0)\ and\ (2,-16)$

5j) Sketch a graph
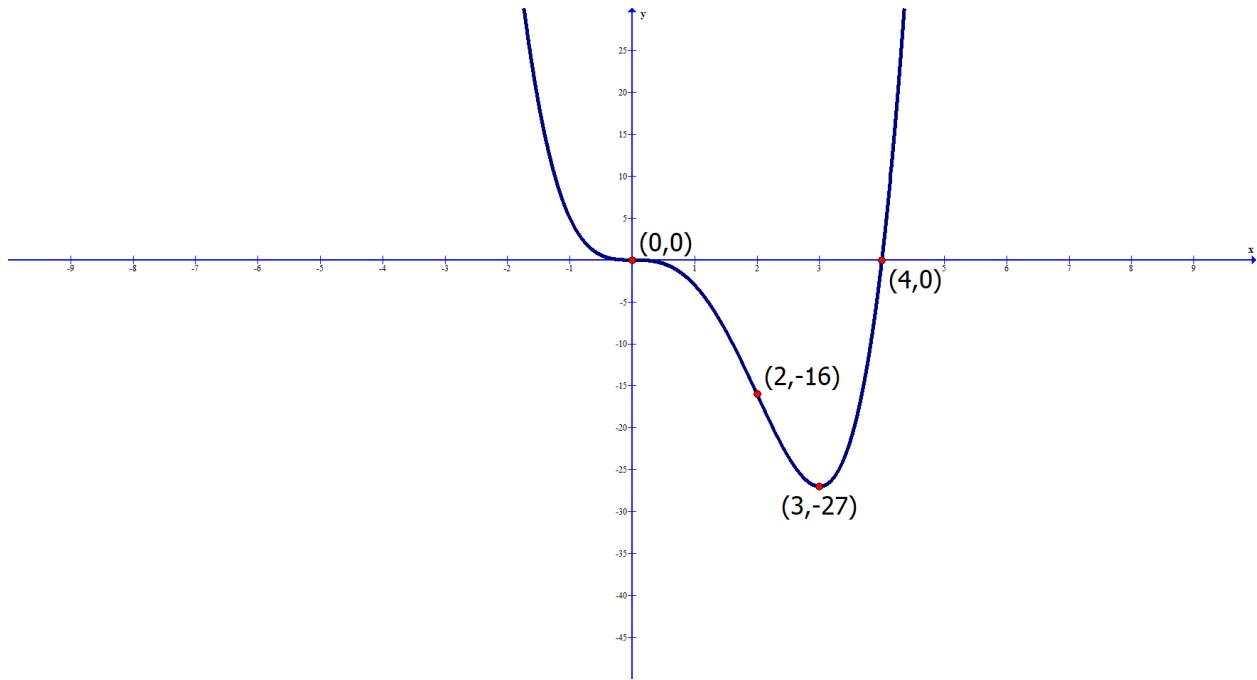
7) $f(x) = \frac{3}{x-4}$ Hint: $f''(x) = \frac{-3}{(x-4)^2}$ $f''(x) = \frac{6}{(x-4)^3}$

7a) Find the domain $X-4=0$ All Real numbers
$X=4$ Except $X=4$

7b) Find the equation of the vertical asymptote   OR $(-\infty, 4) \cup (4, \infty)$

↑ $X=4$

7c) Find the x-intercept(s), if any   None

$3=0$

No Solution

7d) Find the y-intercept, in there is one

$f(0) = \frac{3}{0-4} = \frac{-3}{4}$ $(0, -3/4)$

7e) Find all horizontal asymptotes

Largest power of X only
in the denominator

H.A. $Y=0$

7f) Find the interval(s) where the graph of the function is increasing
7g) Find the interval(s) where the graph of the function is decreasing
7h) Find all relative maxima and
7i) Find all relative minima

$$f'(x) = \frac{-3}{(x-4)^2}$$

$-3 = 0$

No Solution

$(x-4)^2 = 0$

$x = 4$

$-\infty \qquad 4 \qquad \infty$

$(-\infty, 4)$

$x = 0$
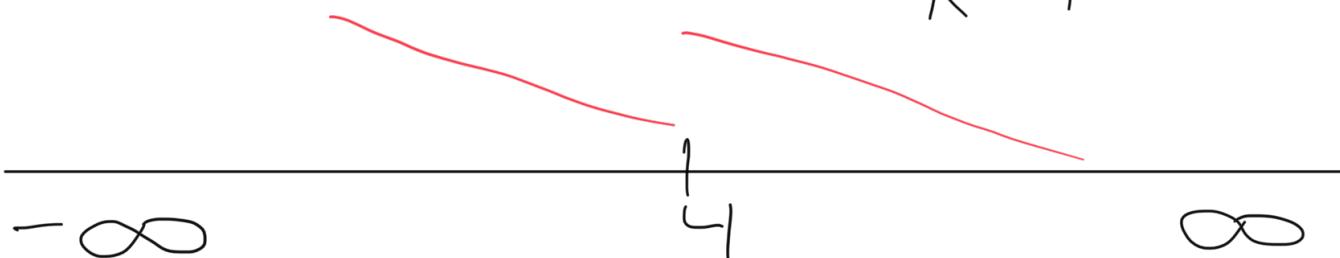
$$f'(0) = \frac{-3}{(0-4)^2} = \frac{-3}{16}$$

Decreasing

$(4, \infty)$

$x = 5$

$$f'(5) = \frac{-3}{(5-4)^2} = -3$$

Decreasing

Never Increases
No max
no min

7j) Find the interval(s) where the graph of the function is concave up (if any)
7k) Find the interval(s) where the graph of the function is concave down (if any)
7l) Find all inflection points (if any)

$$f''(x) = \frac{6}{(x-4)^3}$$

$6 = 0$

NO SOLUTION

$(x-4)^3 = 0$

$x = 4$

$(-\infty, 4)$

$x = 0$

$$f''(0) = \frac{6}{(0-4)^3} = \frac{-6}{64}$$

CONCAVE Down

$(4, \infty)$

$x = 5$

$$f''(5) = \frac{6}{(5-4)^3} = 6$$

CONCAVE UP
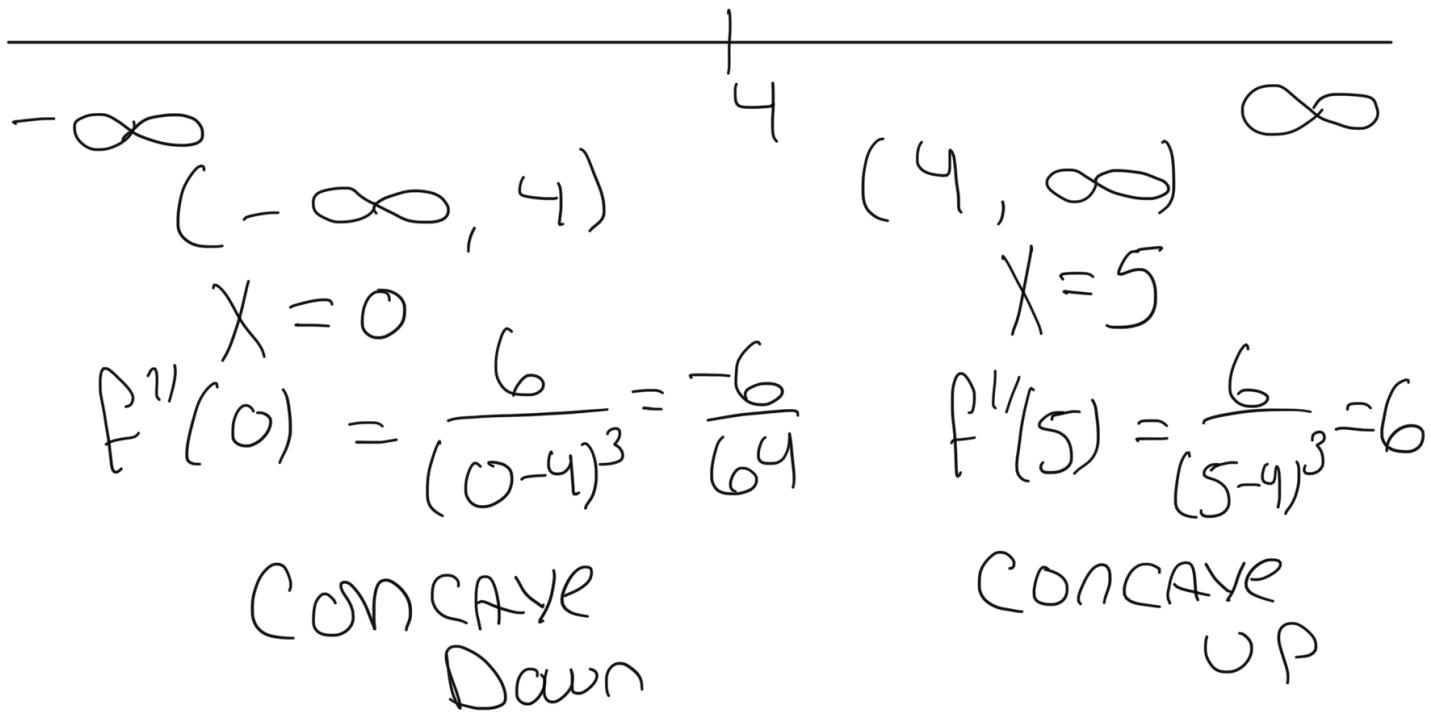
No Inflection POINT aT X=4
Since X= 4 NOT in The
Domain of f(x)

7a) Find the domain $(-\infty, 4) \cup (4, \infty)$ or all real numbers except $x = 4$

7b) Find the equation of the vertical asymptote $x = 4$

7c) Find the x-intercept(s), if any *none*

7d) Find the y-intercept, in there is one $(0, -\frac{3}{4})$

7e) Find all horizontal asymptotes $y = 0$

7f) Find the interval(s) where the graph of the function is increasing Never

7g) Find the interval(s) where the graph of the function is decreasing $(-\infty, 4) \cup (4, \infty)$

7h) Find all relative maxima and   None

7i) Find a ll relative minima   None

7j) Find the interval(s) where the graph of the function is concave up (if any) $(4, \infty)$

7k) Find the interval(s) where the graph of the function is concave down (if any) $(-\infty, 4)$

7l) Find all inflection points (if any) *None*

7m) Sketch a graph

(0,-3/4)
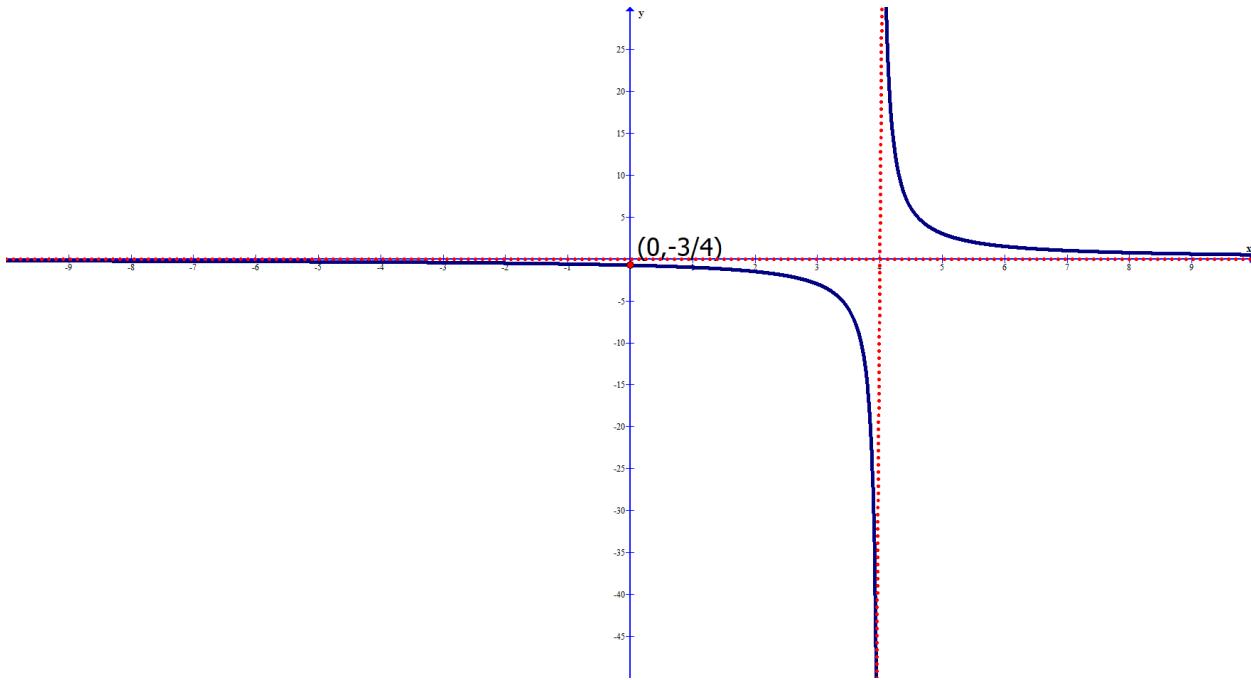
9) $f(x) = \frac{x+2}{x-3}$ Hint: $f''(x) = \frac{-5}{(x-3)^2}$ $f''(x) = \frac{10}{(x-3)^3}$

9a) Find the domain

$$X-3=0$$
$$X=3$$

All Real numbers
Except $X=3$

OR $(-\infty, 3) \cup (3, \infty)$

9b) Find the equation of the vertical asymptote

$$X=3$$

9c) Find the x-intercept(s), if any

$$X+2=0$$
$$X=-2$$

$(-2, 0)$

9d) Find the y-intercept, in there is one

$$f(0) = \frac{0+2}{0-3} = \frac{-2}{3}$$ $(0, \frac{-2}{3})$

9e) Find all horizontal asymptotes

$\frac{1}{1}\frac{x+2}{x-3}$ $y = \frac{1}{1}$ $y = 1$

9f) Find the interval(s) where the graph of the function is increasing
9g) Find the interval(s) where the graph of the function is decreasing
9h) Find all relative maxima and
9i) Find all relative minima

$$f'(x) = \frac{-5}{(x-3)^2}$$

$$-5 = 0$$
no Solution

$$(x-3)^2 = 0$$
$$(x-3)^2 = 0$$
$$x = 3$$

$$-\infty \qquad\qquad 3 \qquad\qquad \infty$$

$$(-\infty, 3) \qquad\qquad [3, \infty)$$

$$x = 2 \qquad\qquad\qquad x = 4$$

$$f'(2) = \frac{-5}{(2-3)^2} = -5 \qquad f'(4) = \frac{-5}{(4-3)^2} = -5$$

Decreasing Decreasing

Never Increases
no max
no min

9j) Find the interval(s) where the graph of the function is concave up (if any)
9k) Find the interval(s) where the graph of the function is concave down (if any)
9l) Find all inflection points (if any)

$$f''(x) = \frac{10}{(x-3)^3}$$

$10 = 0$
No Sol.

$(x-3)^3 = 0$
$x = 3$

$(-\infty, 3) \quad (3, \infty)$

$-\infty \qquad\qquad 3 \qquad\qquad \infty$

$x = 0$

$f''(0) = \frac{10}{(0-3)^3} = -\frac{10}{27}$

Concave Down

$x = 4$

$f''(4) = \frac{10}{(4-3)^3} = 10$

Concave Up
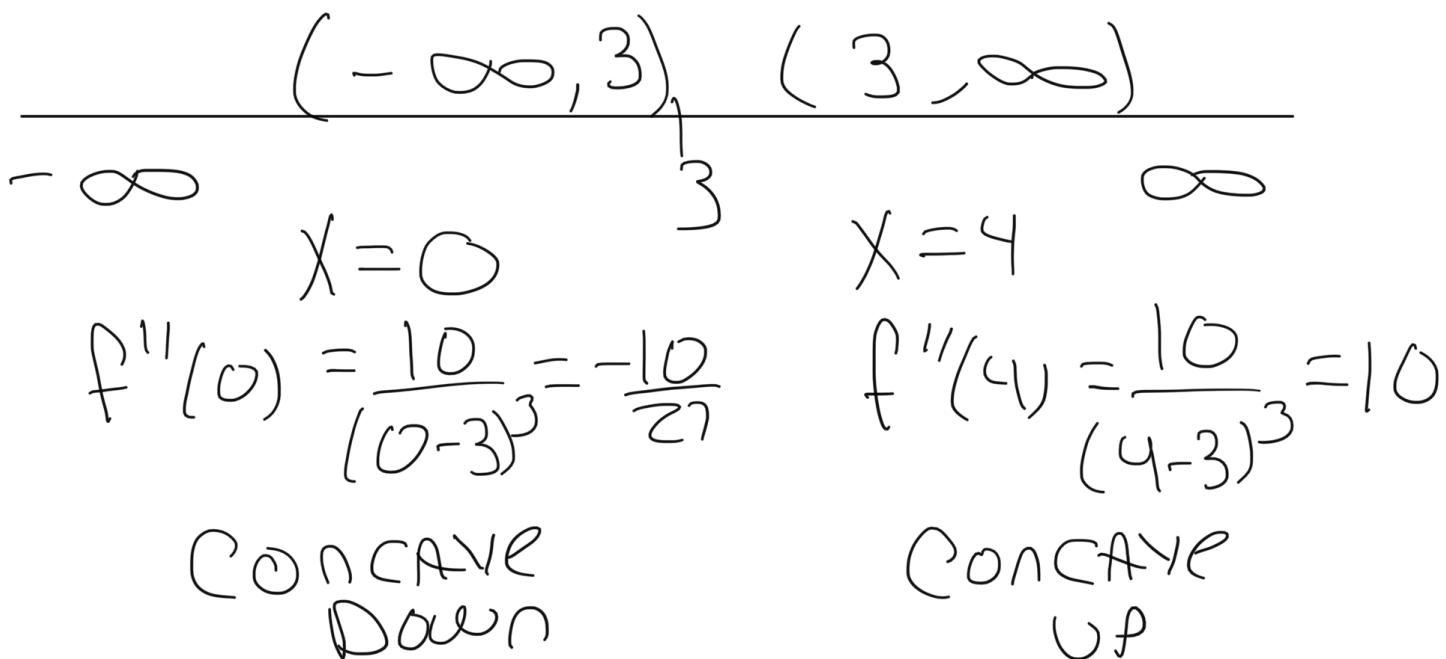
No Inflection point at
$x = 3$ as it is not in
the Domain of $f(x)$

9a) Find the domain $(-\infty, 3) \cup (3, \infty)$ $\;or\; all\; real\; numbers\; except\; x = 3$

9b) Find the equation of the vertical asymptote $\;x = 3$

9c) Find the x-intercept(s), if any $(-2,0)$

9d) Find the y-intercept, in there is one $(0, -\frac{2}{3})$

9e) Find all horizontal asymptotes $\;y = 1$

9f) Find the interval(s) where the graph of the function is increasing Never

9g) Find the interval(s) where the graph of the function is decreasing $(-\infty, 3) \cup (3, \infty)$

9h) Find all relative maxima and None

9i) Find a ll relative minima None

9j) Find the interval(s) where the graph of the function is concave up (if any) $(3, \infty)$

9k) Find the interval(s) where the graph of the function is concave down (if any) $(-\infty, 3)$

9l) Find all inflection points (if any) *None*

9m) Sketch a graph

(-2,0) (0,-2/3)
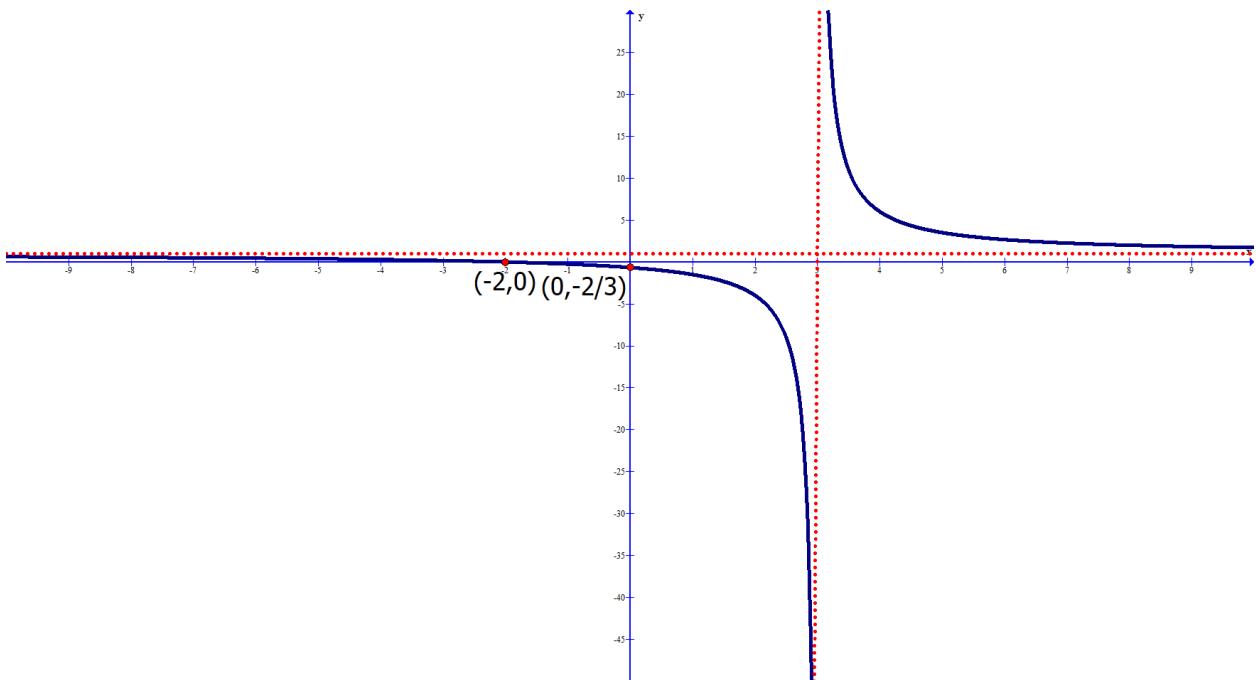
11) $f(x) = xe^x$ Hint: $f'(x) = e^x(x+1)$ $f''(x) = e^x(x+2)$

11a) Find the x-intercept(s), if any

$$xe^x = 0$$

$$x = 0 \qquad e^x = 0$$

$$NO \; Solution$$

11b) Find the y-intercept, in there is one

$$X-INT \; (0,0)$$

$$y = f(0) = 0e^0 = 0$$

$$y-INtercept$$

$$(0,0)$$

11c) Find the interval(s) where the graph of the function is increasing

11d) Find the interval(s) where the graph of the function is decreasing

11e) Find all relative maxima

11f) Find all relative minima

$$f(x) = xe^x$$

$$f'(x) = e^x(x+1)$$

$$e^x = 0$$
No Solution

$$x+1 = 0$$
$$x = -1$$

Min

$-\infty$ ∞

$$(-\infty, -1)^{-1} \quad (-1, \infty)$$

$$x = -2 \qquad\qquad x = 0$$
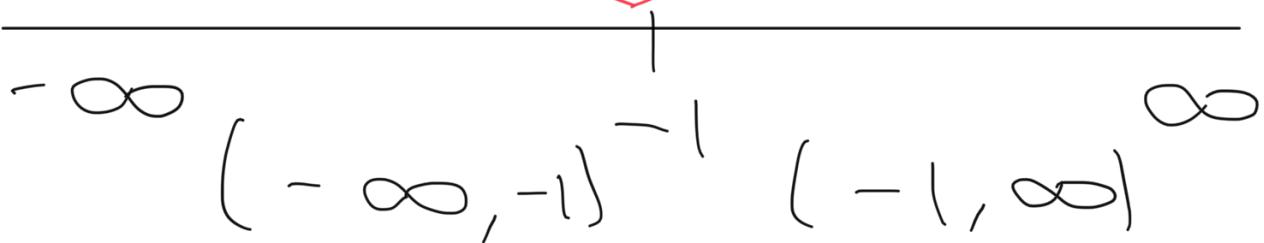
$$f'(-2) = e^{-2}(-2+1) \qquad f'(0) = e^0(0+1)$$
$$= -1/e^2 \qquad\qquad\qquad = 1(1)$$
Decreasing $\qquad\qquad\qquad = 1$
$\qquad\qquad\qquad\qquad\qquad$ Increasing

no max

x-coord min $x = -1$

$$y = f(-1) = -1e^{-1} = -\frac{1}{e^1}$$

$$\text{Min} \left(-1, -\frac{1}{e} \right)$$

11g) Find the interval(s) where the graph of the function is concave up (if any)
11h) Find the interval(s) where the graph of the function is concave down (if any)
11i) Find all inflection points (if any)
11j) Sketch a graph

$$f(x) = xe^x$$
$$f''(x) = e^x(x+2)$$

$$e^x = 0$$
No Solution

$$x+2 = 0$$
$$x = -2$$

$-\infty$ ——————— -2 ——————— ∞

$(-\infty, -2)$

$$x = -3$$
$$f''(-3) = e^{-3}(-3+2)$$
$$= -1/e^3$$
Concave
Down

$(-2, \infty)$

$$x = 0$$
$$f''(0) = e^0(0+2)$$
$$= 2$$
Concave
UP
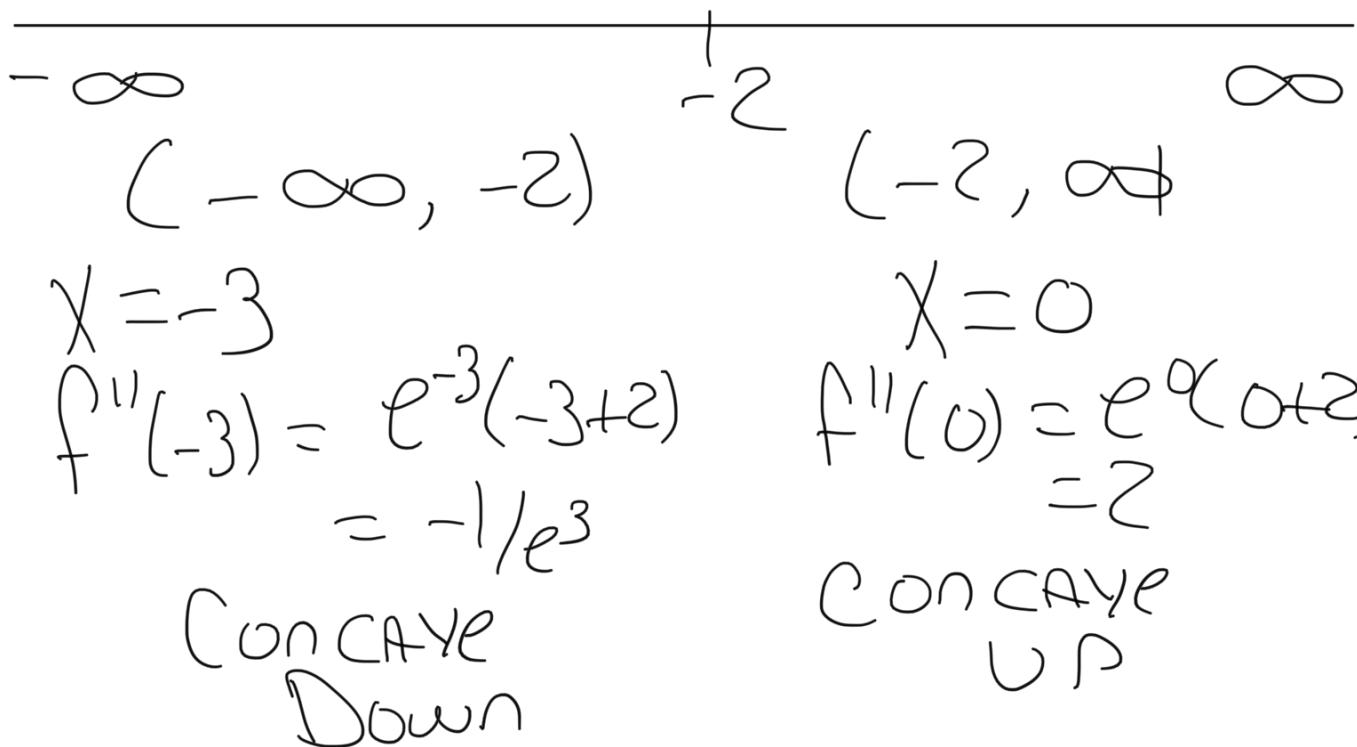
X coordinate of Inflection Point
$$x = -2$$
$$y = f(-2) = -2e^{-2} = \frac{-2}{e^2}$$

Inflection $\left(-2, \frac{-2}{e^2}\right)$
Point

11a) Find the x-intercept(s), if any $(0,0)$

11b) Find the y-intercept, in there is one $(0,0)$

11c) Find the interval(s) where the graph of the function is increasing $(-1, \infty)$

11d) Find the interval(s) where the graph of the function is decreasing $(-\infty, -1)$

11e) Find all relative maxima *none*

11f) Find all relative minima $\left(-1, \frac{-1}{e}\right)$

11g) Find the interval(s) where the graph of the function is concave up (if any) $(-2, \infty)$

11h) Find the interval(s) where the graph of the function is concave down (if any) $(-\infty, -2)$

11i) Find all inflection points (if any) $\left(-2, \frac{-2}{e^2}\right)$

11j) Sketch a graph
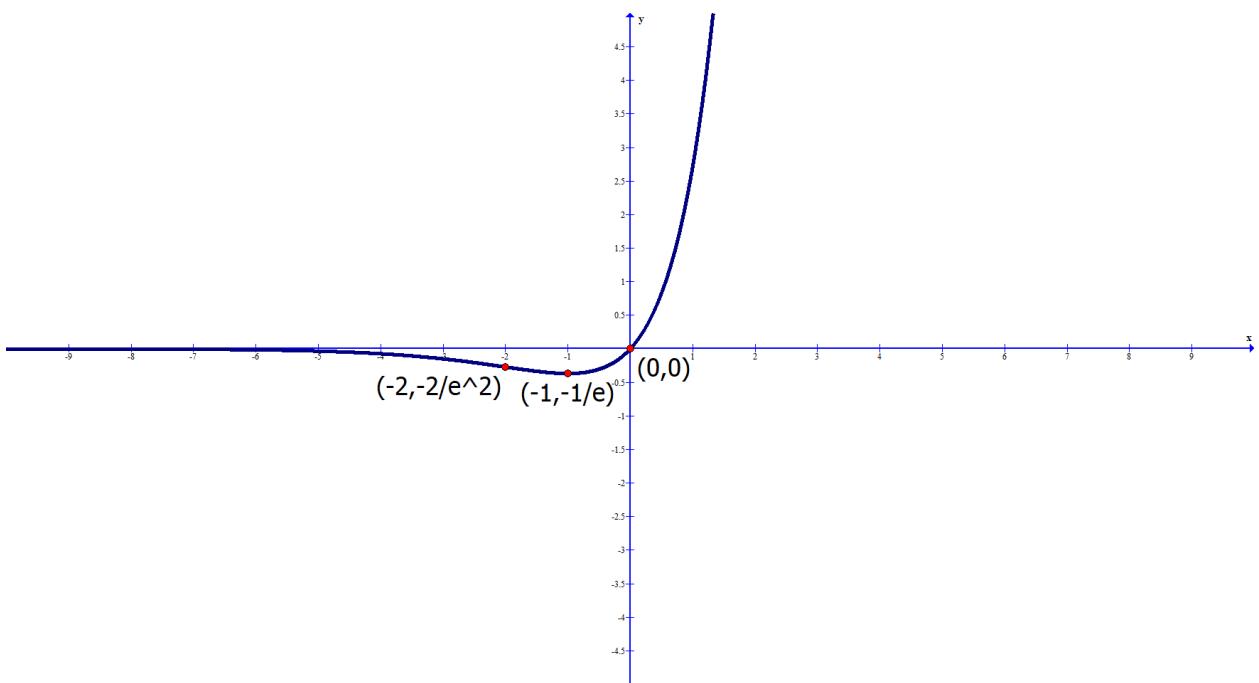
13) $f(x) = 2xe^x$ Hint: $f'(x) = 2e^x(x+1)$ $f''(x) = 2e^x(x+2)$

13a) Find the x-intercept(s), if any

$$\frac{2x}{2} = \frac{0}{2} \qquad e^x = 0$$

$$x = 0 \qquad \text{NO Solution}$$

13b) Find the y-intercept, in there is one

$$y = f(0) = 2(0)e^0 = 0$$

y-intercept

$(0,0)$

x-intercept

$(0,0)$

13c) Find the interval(s) where the graph of the function is increasing
13d) Find the interval(s) where the graph of the function is decreasing
13e) Find all relative maxima
13f) Find all relative minima

$$f(x) = 2xe^x$$

$$f'(x) = 2e^x(x+1)$$

$$2e^x = 0$$
No Solution

$$x+1 = 0$$
$$x = -1$$

min

$$-\infty \qquad\qquad -1 \qquad\qquad \infty$$

$$(-\infty, -1)$$
$$x = -2$$
$$f'(-2) = 2e^2(-2+1)$$
$$= -2/e^2$$
Decreasing

$$(-1, \infty)$$
$$x = 0$$
$$f'(0) = 2e^0(0+1)$$
$$= 2$$
Increasing

No max
x-coord min $x = -1$
$$y = f(-1) = 2(-1)e^{-1} = -2e^{-1}$$
$$= -2/e$$
min $(-1, -2/e)$

13g) Find the interval(s) where the graph of the function is concave up (if any)
13h) Find the interval(s) where the graph of the function is concave down (if any)
13i) Find all inflection points (if any)

$$f(x) = 2xe^x$$

$$f''(x) = 2e^x(x+2)$$

$$2e^x = 0$$

$$x+2 = 0$$

NO Solution

$$x = -2$$

$$-\infty \quad\quad\quad -2 \quad\quad\quad \infty$$

$$(-\infty, -2) \quad\quad (-2, \infty)$$

$$x = -3 \quad\quad\quad\quad x = 0$$

$$f''(-3) = 2e^{-3}(-3+2) \quad\quad f''(0) = 2e^0(0+2)$$

$$= -2/e^3 \quad\quad\quad\quad\quad = 4$$

Concave
Down

Concave
UP
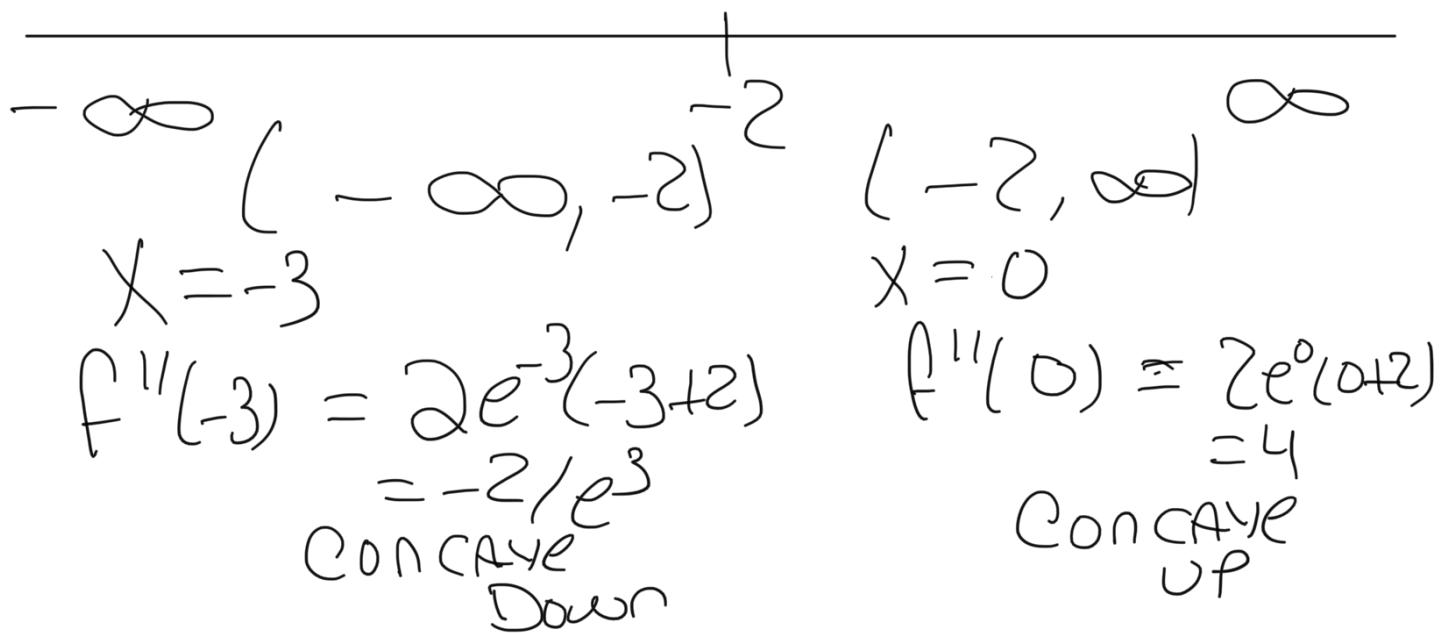
x COORD Inflection point $x = -2$

$$y = f(-2) = 2(-2)e^{-2} = -4e^{-2}$$

$$= -4/e^2$$

Inflection ($-2, -4/e^2$)
Point

13a) Find the x-intercept(s), if any $(0,0)$

13b) Find the y-intercept, in there is one $(0,0)$

13c) Find the interval(s) where the graph of the function is increasing $(-1, \infty)$

13d) Find the interval(s) where the graph of the function is decreasing $(-\infty, -1)$

13e) Find all relative maxima *none*

13f) Find all relative minima $\left(-1, \frac{-2}{e}\right)$

13g) Find the interval(s) where the graph of the function is concave up (if any) $(-2, \infty)$

13h) Find the interval(s) where the graph of the function is concave down (if any) $(-\infty, -2)$

13i) Find all inflection points (if any) $\left(-2, \frac{-4}{e^2}\right)$

13j) Sketch a graph